

# Regularity of the correctors and local gradient estimate of the homogenization for the elliptic equation: linear periodic case

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**Abstract**  $C^\alpha$  and  $W^{1,\infty}$  estimates for the first-order and second-order correctors in the homogenization are presented based on the translation invariant and Li-Vogelius's gradient estimate for the second order linear elliptic equation with piecewise smooth coefficients. If the data are smooth enough, the error of the first-order expansion for piecewise smooth coefficients is locally  $O(\varepsilon)$  in the Hölder norm; it is locally  $O(\varepsilon)$  in  $W^{1,\infty}$  when coefficients are Lipschitz continuous. It can be partly extended to the nonlinear parabolic equation.

**Keywords:** gradient estimate, homogenization, translation invariant, De Giorgi-Nash estimate

**MSC(2000):** 35B27, 35J65

## 1 Introduction

Consider the homogenization of the following elliptic problem: find  $u_\varepsilon \in H_0^1(\Omega)$ ,

$$A_\varepsilon u_\varepsilon \equiv -\frac{\partial}{\partial x_i} \left( a_{ij} \left( \frac{x}{\varepsilon} \right) \frac{\partial u_\varepsilon}{\partial x_j} \right) = f(x), \quad \text{in } \Omega. \quad (1.1)$$

Here  $\Omega \subset \mathbb{R}^n$  is a bounded Lipschitz domain and the summation convention is used.  $A = (a_{ij})$  is symmetric and positive definite;  $a_{ij}(y)$  is 1-periodic in  $y$ ,  $1 \leq i, j \leq n$ ;  $a_{ij}(y)$  is at least piecewise  $C^\mu$  to obtain the error estimate in  $C^\beta, W^{1,\infty}$ .

Assume all of the data are smooth enough, the  $O(\varepsilon)$  error estimate in  $L^\infty$  was presented by A. Bensoussan, J. L. Lions and G. Papanicolaou [1]; also see M. Avellaneda and Lin FangHua [2]. O. A. Oleinik, A. S. Shamaev and G. A. Yosifian [3] proved the  $O(\varepsilon^{1/2})$  estimate in  $H^1$ . Cao and Cui [4] studied the spectral properties and the numerical algorithms in perforated domains. Su *et al* [5] investigated the quasi-periodic problems; Zhang and Cui [6] gave a numerical example for the Rosseland equation. All of these were based on the multiple-scale expansion method [7]. There are also some other famous methods, such as periodic unfolding method [8], Multiscale Finite Element Method(MFEM [9]) and Heterogeneous Multiscale Method(HMM [10]).

The second-order expansion in Section 2 is classical which can be found in Chapter 1 [1] or Chapter 7 [7]. Translation invariant in Section 3 implies the equivalence between the boundary and the interior estimate for an abstract periodic problem. The  $C^\alpha, W^{1,\infty}$  estimates for correctors in Section 4 follow from the De Giorgi-Nash estimate and Li Yanyan-M. Vogelius's work for piecewise smooth coefficients, respectively. In Section 5, more than the traditional  $L^\infty$  estimate ( $a_{ij}(y) \in C^\gamma([0, 1]^n)$ , [2]), we obtain the  $C^\beta$  error estimate ( $a_{ij}(y)$  piecewise  $C^\mu$

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in  $C^{1,\alpha}$  subdomains, Corollary 5.4). At the end, we prove the main result: the error of the first-order expansion is  $O(\varepsilon)$  in  $W_{loc}^{1,\infty}$  for Lipschitz continuous coefficients (Corollary 6.3) based on M. Avellaneda-Lin FangHua's gradient estimate. As far as we know, there are not such kinds  $(C^\beta, W^{1,\infty})$  of error estimates in the homogenization.

## 2 Second-order two-scale expansion

**Definition 2.1.** The periodic cell  $Y = (0, 1)^n$ . Let  $C_{per}^\infty(Y)$  be the subset of  $C^\infty(\mathbb{R}^n)$  of  $Y$ -periodic functions (restricted on  $Y$ ). Denote by  $H_{per}^1(Y)$  the closure of  $C_{per}^\infty(Y)$  in the  $H^1$  norm.  $W_{per}^1(Y) = \{\varphi \in H_{per}^1(Y) : \int_Y \varphi = 0\}$ .  $\|u\|_{W_{per}^1(Y)} \equiv \|\nabla u\|_{L^2(Y)}$ . In the same way, we can define  $W_{per}^1(Y_z)$  where  $Y_z = Y + z, z \in \mathbb{R}^n$ .

If  $u \in H_{per}^1(Y)$ ,  $u$  has the same trace on the opposite faces of  $Y$ . We look for a formal asymptotic expansion of the form

$$u_\varepsilon(x) = u_0(x) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \dots \quad (2.1)$$

where  $u_1(\cdot, y), u_2(\cdot, y)$  are  $Y$ -periodic in  $y$ . Let  $y = \frac{x}{\varepsilon}$ , then

$$\frac{\partial}{\partial x_i} \rightarrow \frac{\partial}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial}{\partial y_i}. \quad (2.2)$$

Substituting (2.1) into (1.1) and equating the power-like terms of  $\varepsilon$ , we introduce  $N_m(y) \in W_{per}^1(Y)$ ,  $1 \leq m \leq n$ , to make the terms of order  $\varepsilon^{-1}$  equal zero,

$$\int_Y a_{ij}(y) \frac{\partial N_m}{\partial y_i} \frac{\partial \varphi}{\partial y_j} = - \int_Y a_{mj}(y) \frac{\partial \varphi}{\partial y_j}, \quad \forall \varphi(y) \in W_{per}^1(Y). \quad (2.3)$$

Then let  $u_1 = N_m \partial_m u_0$ . The problem for  $u_2$  (the part of order  $\varepsilon^0$ ) admits a unique solution if and only if there exists a  $u_0 \in H_0^1(\Omega)$  such that (a compatibility condition, see Theorem 4.26 [7])

$$-\frac{\partial}{\partial x_i} [a_{ij}^0 \frac{\partial u_0}{\partial x_j}] = f, \quad a_{ij}^0 = \int_Y [a_{ij}(y) + a_{il}(y) \frac{\partial N_j}{\partial y_l}] dy. \quad (2.4)$$

This equation called the homogenization equation is well-posed because  $(a_{ij}^0)$  is elliptic (Proposition 6.12 [7]).

Find  $M_{kl} \in W_{per}^1(Y)$ ,  $1 \leq k, l \leq n$ , such that

$$\int_Y a_{ij}(y) \frac{\partial M_{kl}}{\partial y_i} \frac{\partial \varphi}{\partial y_j} = \int_Y \left[ a_{kl} + a_{km} \frac{\partial N_l}{\partial y_m} - a_{kl}^0 \right] \varphi - \int_Y a_{ik} N_l \frac{\partial \varphi}{\partial y_i}, \quad \forall \varphi \in W_{per}^1(Y). \quad (2.5)$$

If  $\varphi(y) = 1$ , the righthand side of the above equation equals zero (a compatibility condition). So let  $u_2 = M_{kl} \partial_{kl}^2 u_0$  to make the  $O(\varepsilon^0)$  terms equal zero. Note that  $N_m, M_{kl}, u_0$  are independent of  $\varepsilon$ . We will use this fact again and again.

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**Corollary 2.2.** Under the hypotheses of Theorem 6.2,

$$\sup_{\Omega'} |\nabla(u_\varepsilon - u_0 - \varepsilon u_1)| \leq C\varepsilon, \quad \Omega' \subset\subset \Omega; \quad (2.6)$$

$$\sup_{\overline{\Omega'}} |A(\frac{x}{\varepsilon}) \nabla(u_\varepsilon - u_0 - \varepsilon u_1)| \leq C\varepsilon, \quad \Omega' \subset \subset \Omega. \quad (2.7)$$

**Proof.** (1) We only need to prove that  $|\varepsilon \partial_i u_2| = |\partial_i(\varepsilon M_{kl}(\frac{x}{\varepsilon}) \partial_{kl}^2 u_0)| \leq C$ .

$$\varepsilon \frac{\partial u_2}{\partial x_i} = \frac{\partial M_{kl}}{\partial y_i}(\frac{x}{\varepsilon}) \partial_{kl}^2 u_0 + \varepsilon M_{kl}(\frac{x}{\varepsilon}) \partial_{ikl}^3 u_0. \quad (2.8)$$

$M_{kl}, \frac{\partial M_{kl}}{\partial y_i}$  are bounded from Theorem 4.4;  $\partial_{ikl}^3 u_0 \in W^{1,q}(\Omega') \hookrightarrow C^{0,\alpha}(\overline{\Omega'}) \subset L^\infty(\Omega')$ .

(2) Note that  $A(\frac{x}{\varepsilon}) = (a_{ij})$  is bounded.  $\square$

**Remark 2.3.** We give the estimate (2.7) because the flux  $(A \nabla u)$  is very important in physics. One can consider the tensor case where the flux may be the stress in linear elasticity.

## 7 Some problems

It is possible to partly extend the above results to the following cases:

(1) tensor case: Avellaneda-Lin's Lemma 6.1 is true for the tensor case [2] and elliptic systems with Neumann boundary conditions [13]; Li-Vogelius's work was extended in [14].

(2) nonlinear case: Fusco and Moscarrello [15] studied the homogenization of quasilinear divergence structure operators. For the second-order expansion, see [16].

(3) parabolic case: the parabolic  $C^{\alpha,\alpha/2}$  estimate under mixed boundary conditions was presented in [17]; Li-Vogelius's gradient estimate was extended to parabolic systems in [14].

(4) nonsmooth case: if the domain is only convex or the righthand side is piecewise smooth, there are many interesting problems. One problem is that the hypotheses in Theorem 5.3 are very strong:  $\partial \Omega \in C^{2,1}$ ,  $f \in W^{1,q}(\Omega)$ ,  $q > n$ . This is a common difficulty for the multiple-scale method (see [1], [2]).

(5) How can we get a global  $W^{1,\infty}$  error estimate with a proper boundary corrector?

Some results will appear elsewhere.

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